

# Stochastic First Integrals, Kernel Functions for Integral Invariants and the Kolmogorov equations

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## Abstract

In this article the authors present stochastic first integrals (SFI), the generalized Itô-Wentzell formula and its application for obtaining the equations for SFI, for kernel functions for integral invariants and the Kolmogorov equations, described by the generalized Itô equations.

*Key words:* stochastic first integrals, stochastic kernel function, stochastic integral invariant, Itô equation, Kolmogorov equations

## Introduction

The concept of a first integral for solution of the deterministic dynamical system is the fundamental concept of an analytical mechanics. For stochastic differential equations (SDE) similar concepts exist as well. They are a *first integral* for Itô's SDE (Doobko, 1978 [1]); a *first forward integral* and a *first backward integral* for Itô's SDE (Krylov and Rozovsky, 1982 [2]); a *stochastic first integral* for generalized Itô's SDE (GSDE) [3, 4]. However a direct transfer of the concept of a first integral from deterministic systems to stochastic systems is impossible.

The different conservation laws, such as energy, weight, impulse, impulse moment, etc., are the basis of invariants and first integrals. For example, if a collection of the enumerable number of the initial solutions for the same dynamical equation is connected to points which are similar to particles, then the number of this points is a conservative value since the conditions of the existence and uniqueness of the solution are fulfilled.

The limiting state of this representation is a density of these points number and conserving of the integral for it into real space. A function which has the same property is called a kernel function for an integral invariant.

By imposing certain restrictions a partial differential equation for the kernel function can be obtained [3, 5, 6]. And we do consider a situation when properties of the functionals which are conserved on some boundary space-time continuum domain can be restrictions. For such cases we refer to representation of local invariants [7]. Evolving structures and functionals connected with initial values domain, and are considered as *dynamical invariants*. For example, an element of a phase-space volume and hypersurfaces are dynamical invariants.

The theory of stochastic integral invariants (SIIs) is one of the approaches for studying stochastic dynamical systems described by the Itô equations [8, 5, 9]. This approach is very effective for defining the conserved functionals for evolving systems, such as: first integrals and stochastic first integrals for the Itô equations (SDE) [1, 10], the length of the random chain [11], a constant velocity of the random walking points [12]. It is applied for constructing the analytical solution of the Langevin type equation [13]; for a determination of probability moments of the point which is walking randomly on a sphere [14, 15]; for obtaining the Itô-Wentzell formula for SDE and the generalized Itô-Wentzell formula for generalized SDE (GSDE) [5, 3, 16, 17]; for forming and solving the problem of the control program with probability one (PCP1) for stochastic systems which are subjected to strong perturbations [18, 19, 20].

We will show an application of the concept of the SFI for forming and proving the theorems about forms of the equations for the SFI. Further, the existence and uniqueness theorems for solutions of GSDE for kernel functions of integral invariants are defined. The important point of this research is the generalized Itô-Wentzell formula. This formula is a differentiation law for compound random function which depends on solutions of GSDE [3, 4, 17]. The choice of such notation comes from the fact that without Poisson perturbations this formula rearranges to the well-known Itô-Wentzell formula [2, 5, 21].

The aim of this article is to demonstrate a possibility of applying the generalized Itô-Wentzell formula for obtaining equations for the SFI, for a complete proof of deriving the equation for stochastic kernel functions (SKFs) of the SIIs and the derivation of the Kolmogorov equations [22].

The article includes three sections. In the first section the following results were obtained: the concept of the SFI is define, the generalized Itô-Wentzell formula is demonstrated and equations for the SFI are given. In the second section we consider a concept of a local stochastic density of a dynamical invariant which is connected with a solution of the GSDE, then we set an equation for it and we establish a link between the local stochastic density and the concept of the SKF for a SII. In the third section we yield the Kolmogorov equation.

# 1. Stochastic first integrals

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a complete probability space,  $\mathcal{F}_t = \{\mathcal{F}_t, t \in [0, T]\}$ ,  $\mathcal{F}_s \subset \mathcal{F}_t$ ,  $s < t$  be a non decreasing flow of the  $\sigma$ -algebras.

Let us consider the next random processes:  $\mathbf{w}(t)$  is the  $m$ -dimensional Wiener process;  $\nu(\Delta t; \Delta \gamma)$  is the standard Poisson measure defined on the space  $[0; T] \times \mathbb{R}^{n'}$  and has properties:  $M[\nu(\Delta t, \Delta \gamma)] = \Delta t \cdot \Pi(\Delta \gamma)$ ;  $\int_{\mathbb{R}(\gamma)} \Pi(d\gamma) < \infty$ ,  $\mathbb{R}(\gamma) = \mathbb{R}^{n'}$  (it means a finiteness of jumps intensity during a small time interval); the one-dimensional processes  $w_k(t)$  and the Poisson measure  $\nu([0; T], \mathcal{A})$  are defined on the same space  $(\Omega, \mathcal{F}, \mathbf{P})$  and  $\mathcal{F}_t$ -measurable for every  $t > 0$  for each set  $\mathcal{A}$  from  $\sigma$ -algebras of the Borel sets and are mutually independent.

Let a random process  $x(t)$  defined on  $\mathbb{R}^n$ , be a solution of the SDE [23]:

$$\begin{aligned} dx_i(t) &= a_i(t)dt + b_{ik}(t)dw_k(t) + \int_{\mathbb{R}(\gamma)} g_i(t; \gamma)\nu(dt; d\gamma), \\ x(t) &= x(t; x_0)|_{t=0} = x_0, \quad \text{для всех } x_0 \in \mathbb{R}^n, \quad i = \overline{1, n}, \quad k = \overline{1, m}, \end{aligned} \quad (1)$$

where  $i = \overline{1, n}$ ,  $k = \overline{1, m}$ ,  $\mathbb{R}(\gamma) = \mathbb{R}^{n'}$  is a space of vectors  $\gamma$ ; and summation is held on the indices double appeared.

The coefficients  $a(t; x)$ ,  $b(t; x)$  and  $g(t; x; \gamma)$  satisfy conditions for the existence and uniqueness of a solution of Eq. (1) (see, for example, [23]).

Following [23] we use a notation  $\int$  instead of  $\int_{\mathbb{R}(\gamma)}$ .

*Definition 1.* [3, 4] Let a random function  $u(t; x; \omega)$  and the solution of Eq. (1) both be defined on the same probability space. The function  $u(t; x; \omega)$  is called a stochastic first integral for GSDE (1) if a condition

$$u(t; x(t; x_0); \omega) = u(0; x_0) \quad (2)$$

is fulfilled with probability one for each solution  $x(t; x_0; \omega)$  of Eq. (1).

Further, we shall not write a parameter  $\omega$  as it is accepted.

Now we represent the generalized Itô-Wentzell formula which we will use for obtaining the equation for the SFI and other results.

**Theorem 1.** (the generalized Itô-Wentzell formula) [6, 17] Let the random process  $x(t) \in \mathbb{R}^n$  be subjected to Eq. (1), and coefficients of this equation satisfy the conditions  $\mathcal{L}_1$ :

$$\mathcal{L}_{1.1}. \quad \int_0^T |a_i(t)|dt < \infty; \quad \int_0^T |b_{ik}(t)|^2 dt < \infty;$$

$$\mathcal{L}_{1.2}. \quad \int_0^T dt \int |g_i(t; \gamma)|^s \Pi(d\gamma) < \infty, \quad s = 1, 2.$$

If  $F(t; x)$ ,  $(t; x) \in [0, T] \times \mathbb{R}^n$  is a scalar function which has the next stochastic differential

$$d_t F(t; x) = Q(t; x)dt + D_k(t; x)dw_k(t) + \int G(t; x; \gamma)\nu(dt; d\gamma) \quad (3)$$

and coefficients of Eq. (3) satisfy conditions  $\mathcal{L}_2$ :

$\mathcal{L}_2.1.$   $Q(t; x)$ ,  $D_k(t; x)$ ,  $G(t; x; \gamma)$  are random functions measurable with respect to  $\mathcal{F}_t$ , which are accorded with the processes  $w_k(t)$ ,  $k = \overline{1, m}$ , and  $\nu(t; \mathcal{A})$  from Eq. (1) for each set  $\mathcal{A} \in \mathfrak{B}$  from the fixed  $\sigma$ -algebra ([23]);

$\mathcal{L}_2.2.$   $Q(t; x) \in \mathcal{C}_{t,x}^{1,2}$ ,  $D_k(t; x) \in \mathcal{C}_{t,x}^{1,2}$ ,  $G(t; x; \gamma) \in \mathcal{C}_{t,x,\gamma}^{1,2,1}$ .

then the next stochastic differential there exists:

$$\begin{aligned} d_t F(t; x(t)) &= Q(t; x(t))dt + D_k(t; x(t))dw_k + \\ &+ \left[ a_i(t) \frac{\partial F(t; x)}{\partial x_i} + \frac{1}{2} b_{ik}(t) b_{j,k}(t) \frac{\partial^2 F(t; x)}{\partial x_i \partial x_j} + \right. \\ &+ b_{ik}(t) \frac{\partial D_k(t; x)}{\partial x_i} \Big|_{x=x(t)} dt + b_{ik}(t) \frac{\partial F(t; x)}{\partial x_i} \Big|_{x=x(t)} dw_k + \\ &+ \int \left[ (F(t; x(t) + g(t; \gamma)) - F(t; x(t))) \right] \nu(dt; d\gamma) + \\ &+ \int G(t; x(t) + g(t; \gamma); \gamma) \nu(dt; d\gamma). \end{aligned} \quad (4)$$

Let us remark that the generalized Itô-Wentzell formula will be applied with extra bounding for coefficients of Eq. (1) [16]:

$$a_i(t) = a_i(t; x) \in \mathcal{C}_{t,x}^{1,2}, \quad b_k(t) = b_k(t; x) \in \mathcal{C}_{t,x}^{1,2}, \quad g_i(t; \gamma) = g_i(t; x; \gamma) \in \mathcal{C}_{t,x,\gamma}^{1,2,1}. \quad (5)$$

For simplicity, we shall use a notation  $\frac{\partial f(t; x(t))}{\partial x_j}$  instead of  $\frac{\partial f(t; x)}{\partial x_j} \Big|_{x=x(t)}$ .

Let us construct an equation for the SFI of the GSDE, taking into account that the stochastic differential for the SFI there exists and has a form:

$$d_t u(t; x) = Q(t; x)dt + D_k(t; x)dw_k(t) + \int G(t; x; \gamma)\nu(dt; d\gamma) \quad (6)$$

Since the definition 1 and representation (6) take place, we apply the generalized Itô-Wentzell formula (4):

$$\begin{aligned} d_t u(t; x(t)) &= Q(t; x(t))dt + D_k(t; x(t))dw_k + b_{ik}(t) \frac{\partial}{\partial x_i} u(t; x(t))dw_k + \\ &+ \left[ a_i(t) \frac{\partial}{\partial x_i} u(t; x(t)) + \frac{1}{2} b_{ik}(t) b_{j,k}(t) \frac{\partial^2 u(t; x(t))}{\partial x_i \partial x_j} + \right. \\ &+ b_{ik}(t) \frac{\partial}{\partial x_i} D_k(t; x(t)) \Big] dt + \int G(t; x(t) + g(t; \gamma); \gamma) \nu(dt; d\gamma) + \\ &+ \int [(u(t; x(t) + g(t; \gamma)) - u(t; x(t)))] \nu(dt; d\gamma) = 0. \end{aligned} \quad (7)$$

The appearance of equation for the SFI depends if the function  $g(t; \gamma)$  depends or does not depend on  $x$ . Let us consider both situations.

Let function  $g(t; \gamma)$  be not depend on  $x$ , and coefficients of Eq. (6) are:

$$Q(t; x) = -\left[a_i(t) \frac{\partial}{\partial x_i} u(t; x) + \frac{1}{2} b_{ik}(t) b_{j,k}(t) \frac{\partial^2 u(t; x)}{\partial x_i \partial x_j} + b_{ik}(t) \frac{\partial}{\partial x_i} (b_{j,k}(t) \frac{\partial}{\partial x_j} u(t; x))\right], \quad (8a)$$

$$D_k(t; x) = -b_{ik}(t) \frac{\partial}{\partial x_i} u(t; x), \quad (8b)$$

$$G(t; x; \gamma) = u(t; x - g(t; \gamma)) - u(x; t). \quad (8c)$$

**Theorem 2.** *In order for the random function  $u(t, x) \in \mathcal{C}_{t,x}^{1,2}$  having the stochastic differential, appearing as Eq. (6) to be the SFI for the system (1), it is sufficient that this function would be the solution of Eq. (1) with coefficients (8).*

*Proof* The sufficiency of it connects with the test of truth for Eq. (2). It is fulfilled if the coefficients of Eq. (6) are determined by (8). This fact is tested by substitution (7) into (8).  $\diamond$

Assume that  $g(t; \gamma) = g(t; x; \gamma)$ . Then:

$$dx(t) = a(t)dt + b_k(t)dw_k(t) + \int g(t; x(t); \gamma) \nu(dt; d\gamma), \quad (9)$$

where  $a(t) = (a_i(t))$ ,  $b_k(t) = (b_{ik}(t))$ ,  $g(\cdot) = (g_i(\cdot))$ ,  $i = \overline{1, n}$ ,  $k = \overline{1, m}$ .

Let us introduce new functions and make a few remarks about their properties.

If we have  $y = x + g(t; x; \gamma)$  in Eq. (7), then denote its solution with respect to  $x$  as  $x^{-1}(t; y; \gamma)$ .

If we consider the domain of the uniqueness, then we can choose

$$y = z + g(t; z; \gamma), \quad (10)$$

from this domain, and we get the next equality:

$$x^{-1}(t; z + g(t; z; \gamma); \gamma) = z. \quad (11)$$

Let  $u(t; x)$  be a random function, and its stochastic differential defined by Eq. (6), and coefficients of Eq. (6) are defined by (8a), (8b) and

$$G(t; x; \gamma) = u(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) - u(x; t). \quad (12)$$

**Theorem 3.** *The random function  $u(t; x) \in \mathcal{C}_{t,x}^{1,2}$ , having the stochastic differential, appeared as Eq. (6) with coefficients  $D_k(t; x)$ ,  $Q(t; x)$  and  $G(t; x; \gamma)$  of Eq. (6) defined by (8a), (8b) and (12) respectively, is the the SFI for the system (9). Moreover, these conditions are necessary and sufficient, if the given bounding for coefficients is set.*

*Proof* Sufficiency is based on the fact, that a multiplier for the Poisson measure in Eq. (7) must equal to zero. Taking into account Eq. (12) and properties (10) and (11), we have the next equality:

$$\begin{aligned}
& \int G(t; x(t) + g(t; x(t); \gamma)) \nu(dt; d\gamma) + \\
& + \int [(u(t; x(t) + g(t; x(t); \gamma)) - u(t; x(t)))] \nu(dt; d\gamma) = \\
& = \int [(u(t; x(t) + g(t; x(t); \gamma) - g(t; x^{-1}(t; x + g(t; x(t); \gamma); \gamma) - \\
& \quad - u(x(t) + g(t; x(t); \gamma); t))] \nu(dt; d\gamma) + \\
& + \int [(u(t; x(t) + g(t; x(t); \gamma)) - u(t; x(t)))] \nu(dt; d\gamma) \equiv 0.
\end{aligned} \tag{13}$$

Sufficiency is proved.

Necessity follows from conditions for existence and uniqueness for solutions of Eq. (9). Then we have the uniqueness of the appearance for the stochastic differential  $d_t F(t; x(t))$  for the function  $F(t; x)$  which has the stochastic differential (2).

Proof is complete.  $\diamond$

## 2. Local stochastic density, kernel function for integral invariants for the Itô equations and their equations

Let  $x(t)$  be a random process defined on  $\mathbb{R}^n$  and it is a solution of the next equation

$$\begin{aligned}
dx_i(t) &= a_i(t; x(t))dt + b_{ik}(t; x(t))dw_k(t) + \int g_i(t; x(t); \gamma) \nu(dt; d\gamma), \\
x(t) &= x(t; x_0)|_{t=0} = x_0, \quad \text{for every } x_0 \in \mathbb{R}^n, \quad i = \overline{1, n}, \quad k = \overline{1, m},
\end{aligned} \tag{14}$$

and its coefficients satisfy more rigid conditions, then we used before [?, c. 278–290, 298–302]:

$$a_i(t; x) \in \mathcal{C}_{t,x}^{1,1}, \quad b_{ik}(t; x) \in \mathcal{C}_{t,x}^{1,2}; \tag{15}$$

and

$$\int_0^T dt \int |\nabla^\alpha g(t; x; \gamma)|^\beta \Pi(d\gamma) < \infty, \quad \alpha = 1, 2, \quad \beta = \overline{1, 4}, \tag{16}$$

where  $\nabla^k$  denotes all possible varieties of combinations of  $k$ –th partial derivatives with respect to  $x$ , taking into account a continuity of there derivatives.

Conditions (15) and (16) are sufficient for existence and uniqueness of Eq. (14) [23].

Let us introduce a few definitions.

*Definition 2.* [24] Let  $S(t) = S(t; v)$ ,  $v \in \hat{\mathcal{F}} \subset \mathcal{F}$  be measurable mapping from  $\mathbb{R}^n$  into  $\mathbb{R}^n$ . The set  $\Delta(t) = S(t; v) \cdot \Delta(0)$  is the dynamical invariant of the domain  $\Delta(0)$  for the process  $x(t) \in \mathbb{R}^n$ , if the following holds:

$$\mathbf{P}\left\{x(t) \in \Delta(t) \mid x = x_0\right\} = 1, \quad \text{for every } t > 0, \quad \text{for every } x \in \Delta(0).$$

Let  $\rho(t; x; \omega)$  be a random function, which is  $\mathcal{F}_t$ -measurable. Further, we shall not write a parameter  $\omega$  as we said before.

*Definition 3.* A random function  $\rho(t; x)$  is a stochastic density of the dynamic invariant for the equation connected with Eq. (14), if a random function  $u(t; J; x) = J\rho(t; x)$  is the SFI, and it satisfies the next equality

$$J(t; x_0)\rho(t; x(t; x_0)) = \rho(0; x_0), \quad \text{for every } x_0 \in \Gamma \subset \mathbb{R}^n, \quad (17)$$

where function  $J(t) = J(t, x_0)$  is a solution of equation

$$\begin{aligned} dJ(t) &= J(t) \left\{ K(t)dt + \frac{\partial b_{ik}(t; x(t))}{\partial x_i} dw_k(t) + \right. \\ &+ \left. \int \left( \det \left[ A(\delta_{i,j} + \frac{\partial g_i(t; x(t); \gamma)}{\partial x_j}) \right] - 1 \right) \nu(dt, d\gamma) \right\}, \\ J(t) &= J(t, x_0) \Big|_{t=0} = 1, \quad \dim A(\cdot) = n \times n; \end{aligned} \quad (18)$$

$\delta_{i,j}$  is Kronecker symbol;

$$\begin{aligned} K(t) &= \left[ \frac{\partial a_i(t; x(t))}{\partial x_i} + \right. \\ &+ \left. \frac{1}{2} \left( \frac{\partial b_{ik}(t; x(t))}{\partial x_i} \cdot \frac{\partial b_{jk}(t; x(t))}{\partial x_j} - \frac{\partial b_{ik}(t; x(t))}{\partial x_j} \cdot \frac{\partial b_{jk}(t; x(t))}{\partial x_i} \right) \right]. \end{aligned}$$

and  $x(t)$  is a solution of Eq. (14).

The equation (18) is the one for a Jacobian of transformation from  $x_0$  to  $x(t; x_0)$ . The solution of Eq. (18) can be represented in the form:

$$\begin{aligned} J(t) &= \exp \left\{ \int_0^t \left[ K(\tau) - \frac{1}{2} \left( \frac{\partial b_{ik}(\tau; x(\tau))}{\partial x_i} \right)^2 \right] d\tau + \int_0^t \frac{\partial b_{ik}(\tau; x(\tau))}{\partial x_i} dw_k(\tau) + \right. \\ &+ \left. \int_0^t \int \ln \left| \det \left[ A(\delta_{i,j} + \frac{\partial g_i(\tau; x(\tau); \gamma)}{\partial x_j}) \right] \right| \nu(d\tau, d\gamma) \right\}. \end{aligned}$$

Note that  $J(t) > 0$  for every  $t \geq 0$ .

Now we define an appearance of a partial differential equation for the function  $\rho(t; x)$ . Following Eq. (17), we get

$$d_t J(t; x_0)\rho(t; x(t; x_0)) = 0, \quad \text{for every } x_0 \in \Gamma \subset \mathbb{R}^n \quad (19)$$

Assume, the function  $\rho(t; x)$  is a solution of this equation:

$$\begin{aligned} d_t \rho(t; x) &= Q(t; x)dt + D_k(t; x)dw_k(t) + \int G(t; x; \gamma)\nu(dt; d\gamma), \\ \rho(t; x)|_{t=0} &= \rho(x) \in C_0^2. \end{aligned} \quad (20)$$

Here the function  $\rho(x)$  is an initial condition. The coefficients of this equation satisfy conditions

$$Q(t; x) \in \mathcal{C}_{t,x}^{1,1}, \quad D_k(t; x) \in \mathcal{C}_{t,x}^{1,2}; \quad (21)$$

and

$$\int_0^T dt \int |\nabla^\alpha G(t; x; \gamma)|^\beta \Pi(d\gamma) < \infty, \quad \alpha = 1, 2, \quad \beta = \overline{1, 4}. \quad (22)$$

Now we show that the function  $\rho(t; x)$  would be a solution for the next equation which we obtained before in [17] by using another approach:

$$\begin{aligned} d_t \rho(t; x) &= - \left[ \frac{\partial \rho(t; x) a_i(t; x)}{\partial x_i} - \frac{\partial^2 \rho(t; x) b_{ik}(t; x) b_{j,k}(t; x)}{\partial x_i \partial x_j} \right] dt - \\ &\quad - \frac{\partial \rho(t; x) b_{ik}(t; x)}{\partial x_i} dw_k(t) + \\ &+ \int \left[ \rho(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) \bar{D}(x^{-1}(t; x; \gamma)) - \rho(t; x) \right] \nu(dt; d\gamma), \\ \rho(t; x)|_{t=0} &= \rho(x) \in C_0^2; \quad \rho(x) \geq 0, \end{aligned} \quad (23)$$

where  $x^{-1}(t; x; \gamma)$  is defined as a solution of equation

$$y + g(t; y; \gamma) = x \quad (24)$$

with respect to  $y$  for every domain of uniqueness for this solution; and  $\bar{D}(x^{-1}(t; x; \gamma))$  denotes a Jacobian of transformation which corresponds to this substitution. Further, we will denote matrices' determinants as letters with bars (i.e.,  $\bar{A}$ ,  $\bar{D}$ ).

Since the stochastic differential – i. e. the GSDE – can be represented as a sum of the two paths, where the first addend is defined by the Wiener process, and the second one is defined by the Poisson jumps, hence, we rewrite equality (19) as two equalities together:

$$[d_t J(t; x_0) \rho(t; x(t; x_0))]_1 = 0, \quad \text{for every } x_0 \in \Gamma, \quad (25)$$

$$[d_t J(t; x_0) \rho(t; x(t; x_0))]_2 = 0, \quad \text{for every } x_0 \in \Gamma, \quad (26)$$

where (25) corresponds to Itô's stochastic differential, and (26) belongs to the Poisson part.

**Theorem 4.** Assume that the coefficients of Eq. (14) and Eq. (20) satisfy terms (15), (16) and (21), (22), respectively. Then the coefficients  $Q(t; x)$ ,  $D_k(t; x)$ ,  $G(t; x; \gamma)$  of Eq. (20) providing condition (19) are uniquely determined by the next equalities



$$I.1. -Q(t; x) = \frac{\partial \rho(t; x) a_i(t; x)}{\partial x_i} - \frac{\partial^2 \rho(t; x) b_{ik}(t; x) b_{j,k}(t; x)}{\partial x_i \partial x_j},$$

$$I.2. -D_k(t; x) = \frac{\partial \rho(t; x) b_{ik}(t; x)}{\partial x_i};$$

$$I.3. G(t; x; \gamma) = \rho(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) \bar{D}(x^{-1}(t; x; \gamma)) - \rho(t; x).$$

Moreover, if we have initial conditions for Eq. (23), then the function  $\rho(t; x)$  is the unique solution of Eq. (20) with coefficients defined by I.1, I.2.

*Proof* The bounding of the theorem for the coefficients of Eq. (20) provides an application of the Generalized Itô – Wentzell formula (4). Therefore, we get:

$$\begin{aligned} d_t \rho(t; x(t)) = & Q(t; x(t)) dt + D_k(t; x(t)) dw_k + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} \rho(t; x(t)) dw_k + \\ & + \left[ a_i(t; x(t)) \frac{\partial}{\partial x_i} \rho(t; x(t)) + \frac{1}{2} b_{ik}(t; x(t)) b_{j,k}(t; x(t)) \frac{\partial^2 \rho(t; x(t))}{\partial x_i \partial x_j} + \right. \\ & \left. + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} D_k(t; x(t)) \right] dt + \int G(t; x(t) + g(t; x(t); \gamma)) \nu(dt; d\gamma) + \\ & + \int \left[ (\rho(t; x(t) + g(t; x(t); \gamma)) - \rho(t; x(t))) \right] \nu(dt; d\gamma). \end{aligned} \quad (27)$$

Let us consider the components of Itô's differential for the above expression:

$$\begin{aligned} [d_t \rho(t; x(t))]_1 = & Q(t; x(t)) dt + D_k(t; x(t)) dw_k(t) + \\ & + \left[ a_i(t; x(t)) \frac{\partial}{\partial x_i} \rho(t; x(t)) + \frac{1}{2} b_{ik}(t; x(t)) b_{j,k}(t; x(t)) \frac{\partial^2 \rho(t; x(t))}{\partial x_i \partial x_j} + \right. \\ & \left. + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} D_k(t; x(t)) \right] dt + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} \rho(t; x(t)) dw_k(t). \end{aligned} \quad (28)$$

An appearance of  $[d_t \rho(t; x(t))]_2$  we will research later.

By using (18), (28) and Itô's formula [23], we obtain:

$$\begin{aligned} [d_t J(t) \rho(t; x(t))]_1 = & \rho(t; x(t)) [dJ(t)]_1 + J(t) [d\rho(t; x(t))]_1 + \\ & + J(t) b_{ik}(t; x(t)) \frac{\partial b_{j,k}(t; x(t))}{\partial x_j} \frac{\partial \rho(t; x(t))}{\partial x_i} dt = \\ = & J(t) \left[ b_{ik}(t; x(t)) \frac{\partial b_{j,k}(t; x(t))}{\partial x_j} \frac{\partial \rho(t; x(t))}{\partial x_i} + a_i(t; x(t)) \frac{\partial}{\partial x_i} \rho(t; x(t)) + \right. \\ & + \frac{1}{2} b_{ik}(t; x(t)) b_{j,k}(t; x(t)) \frac{\partial^2 \rho(t; x(t))}{\partial x_i \partial x_j} + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} D_k(t; x(t)) + \\ & + b_{ik}(t; x(t)) \frac{\partial}{\partial x_i} D_k(t; x(t)) + Q(t; x(t)) + \\ & + \rho(t; x(t)) \frac{\partial a_i(t; x(t))}{\partial x_i} + \frac{1}{2} \left[ \frac{\partial b_{ik}(t; x(t))}{\partial x_i} \frac{\partial b_{j,k}(t; x(t))}{\partial x_j} - \frac{\partial b_{ik}(t; x(t))}{\partial x_j} \frac{\partial b_{ik}(t; x(t))}{\partial x_i} \right] \Big] dt + \\ & + J(t) \left[ b_{ik}(t; x(t)) \frac{\partial \rho(t; x(t))}{\partial x_i} + \rho(t; x(t)) \frac{\partial b_{ik}(t; x(t))}{\partial x_i} + D_k(t; x(t)) \right] dw_k(t). \end{aligned}$$

Since (25) holds, we get the necessity of the theorem's conditions I.1 and I.2.

If we have the given restrictions, then solutions of equations (20), (23) and (28) exist and they are unique [2].

Let us research the statement I.3 of this theorem.

Between the neighboring Poisson jumps, the random process is subjected to Wiener's perturbations only, and an order of smoothness for function  $\rho(t; x(t))$  is conserved. At the time of the jumps the next functional is added:

$$\begin{aligned} \mathcal{I}(t) = & \int_0^t \int G(\tau; x(\tau) + g(\tau; x(\tau); \gamma)) \nu(d\tau; d\gamma) + \\ & + \int_0^t \int [(\rho(\tau; x(\tau) + g(t; x(t); \gamma)) - \rho(\tau; x(\tau))) \nu(d\tau; d\gamma) \end{aligned} \quad (29)$$

and it leaves the smoothness property invariant.

Following theorems about properties for a partial differentiation equation [2], we set, that the solution of this does exist on the next interval between jumps, and the order of its smoothness is conserved.

As (26) holds, then we need to test that from the following expressions

$$\begin{aligned} [d_t \rho(t; x(t))]_2 = & \int \left\{ (\rho(t; x(t) + g(t; x(t); \gamma)) - \rho(t; x(t)) + \right. \\ & + [\rho(t; x(t) + g(t; x(t); \gamma) - g(t; x^{-1}(t; x(t) + g(t; x(t); \gamma); \gamma); \gamma)]) \times \\ & \times \bar{D}(x^{-1}(t; x(t) + g(t; x(t); \gamma))) - \rho(t; x(t) + g(t; x(t); \gamma)) \left. \right\} \nu(dt; d\gamma) = \\ = & - \int [\rho(t; x(t)) - \rho(t; x(t)) \bar{D}(\frac{\partial x_i^{-1}(t; x + g(t; x; \gamma); \gamma)}{\partial(x_j + g_j(t; x; \gamma))})] \nu(dt; d\gamma), \\ [dJ(t)]_2 = & J(t) \int \left( \det \left[ A(\delta_{i,j} + \frac{\partial g_i(t; x(t); \gamma)}{\partial x_j}) \right] - 1 \right) \nu(dt, d\gamma) \end{aligned} \quad (30)$$

the next differential equals zero:

$$\begin{aligned} [d_t J(t) \rho(t; x(t))]_2 = & \\ = & \int \left\{ [\rho(t; x(t)) + (\rho(t; x(t)) \bar{D}(\frac{\partial x_j^{-1}(t; x + g(t; x; \gamma); \gamma)}{\partial(x_i + g_i(t; x; \gamma))}) - \rho(t; x(t))] \times \right. \\ & \times [J(t) + J(t) \bar{A}(\delta_{i,j}(t) + \frac{\partial g_i(t; x(t); \gamma)}{\partial x_l}) - J(t)] - J(t) \rho(t; x(t)) \left. \right\} \nu(dt; d\gamma) = \\ = & J(t) \int [\rho(t; x(t)) \bar{D}(\frac{\partial x_j^{-1}(t; x(t) + g(t; x; \gamma); \gamma)}{\partial(x_i + g_i(t; x; \gamma))}) \bar{A}(\frac{\partial(g_i(t; x(t); \gamma) + x_i)}{\partial x_j}) - \\ & - \rho(t; x(t))] \nu(dt; d\gamma). \end{aligned}$$

In fact, by using Eq. (33), Eq. (34) for the function  $\rho(t; x(t))$ , we substitute the respective expression to Eq. (29) in accordance with (24).

Hence, if the expression under integral sign equals zero, then the condition I.3 is fulfilled. This is true, if  $\bar{D}(\cdot) \bar{A}(\cdot) = 1$ .

Taking into account the definition (24) and representation  $x^{-1}(t; x + g(t; x; \gamma)) = x$ , we obtain:

$$\begin{aligned} \det \left[ D \left( \frac{\partial x_i^{-1}(t; x + g(t; x; \gamma); \gamma)}{\partial(x_l + g_l(t; x; \gamma))} \right) A \left( \frac{\partial(g_l(t; x; \gamma) + x_l)}{\partial x_j} \right) \right] &= \\ &= \det S \left( \frac{\partial x_i^{-1}(t; x + g(t; x; \gamma); \gamma)}{\partial(x_l + g_l(t; x; \gamma))} \frac{\partial(g_l(t; x; \gamma) + x_l)}{\partial x_j} \right) = \\ &= \det S \left( \frac{\partial x_i^{-1}(t; x + g(t; x; \gamma); \gamma)}{\partial x_j} \right) = \det S(\delta_{i,j}) = 1. \end{aligned}$$

It leads to the next statement: if we have domains, where one-by-one mapping between variables from equation  $y = x + g(t; x; \gamma)$  there exists, then the solution of the equation from this theorem there exists and it is unique.

Proof is complete.

If a requirement for existence and uniqueness of the solution of Eq. (14) is fulfilled in the real space, then the next conditions are added [3, 5]:

$$\int_{\mathbb{R}^n} f(x) \rho(t; x) d\Gamma(x) = \int_{\mathbb{R}^n} f(x(t; y)) \rho(y) d\Gamma(y), \quad \int_{\mathbb{R}^n} \rho(t; x) d\Gamma(x) = 1 \quad (31)$$

for every continuous and bounded function  $f(x)$ , where a function  $\rho(x) = \rho(0; x)$  satisfies terms:

$$\lim_{|x| \rightarrow \infty} \frac{\partial^k \rho(t; x)}{\partial x_i^k} \Big|_{t=0} = 0, \quad k = 0, 1, 2. \quad i = \overline{1, n}.$$

The relation (31) is a definition of a stochastic kernel function for SII [3, 17].

Let us note, that a full collection of kernel functions for SII  $\rho_r(t; x)$ ,  $r = \overline{1, n+1}$  exists, similar to deterministic systems. As it is well known, a full collection of kernel functions of  $n$ -th order integral invariants (nII) is a collection of the kernels, if another function which is a kernel function of nII, would be represented as a function of those kernels. Each kernel  $\rho(t; x)$  is uniquely determined by an initial value  $\rho(x)$  and the full collection of kernels [4, 5, 25]. For the stochastic processes a proof scheme of this result is similar to a proof scheme for deterministic systems.

It is necessary to stress that existence of the density  $\rho(t; x)$  connects with properties of a measure  $\mu(t; \Delta)$  ( $\int_{\mathbb{R}^n} \mu(t; d(\Delta)) = \text{const}$ ) which is produced by some mapping. If a limit of  $\mu(t; \Delta)/\mu(\Delta)$  as  $\Delta \downarrow 0$  exists there in some sense, then it is identified with the function  $\rho(t; x)$ . One must take into account that the function  $\rho(t; x)$  may not have a necessary smoothness which allows to construct Eq. (23).

### 3. An application of the equations for the stochastic density function for obtaining the Kolmogorov equations

Assume,  $x(t)$  is a solution of the next system (14) with coefficients  $a_i(t; x)$ ,  $b_{ik}(t; x)$  and  $g(t; x; \gamma)$  which are satisfied conditions (15) and (16). These conditions provide the existence and uniqueness for solution of Eq. (14) under some terms for equation which we consider below.

Let a random non-negative function  $\rho(t; x)$  and random process  $x(t)$  be determined mutually with respect to  $\mathcal{F}_t$ . The function  $\rho(t; x)$  is the stochastic density function for the integral invariant which is connected with  $x(t)$  as far as it satisfies conditions (31).

An equation for function  $\rho(t; x)$  we could obtain applying different approaches. Now we use Eq. (23).

Further, we use the centered Poisson measure  $\tilde{\nu}(\Delta t, \Delta \gamma) = \nu(\Delta t, \Delta \gamma) - \Delta t \Pi(\Delta \gamma)$  for Eq. (23):

$$\begin{aligned} d_t \rho(t; x) = & -\frac{\partial \rho(t; x) b_{i,k}(t; x)}{\partial x_i} dw_k(t) - \left[ \frac{\partial (\rho(t; x) a_i(t; x))}{\partial x_i} - \right. \\ & \left. - \frac{1}{2} \frac{\partial^2 (\rho(t; x) b_{i,k}(t; x) b_{j,k}(t; x))}{\partial x_i \partial x_j} \right] dt + \\ & + \int [\rho(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) \bar{D}(x^{-1}(t; x; \gamma)) - \rho(t; x)] \Pi(d\gamma) dt + \\ & + \int [\rho(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) \bar{D}(x^{-1}(t; x; \gamma)) - \rho(t; x)] \tilde{\nu}(dt; d\gamma). \end{aligned} \quad (32)$$

Let us introduce a notation:  $\mathbf{M}[\rho(t; x)] = p(t; x)$ .

After calculating the mean function for Eq. (32) we get that the function  $p(t; x)$  satisfies an equation

$$\begin{aligned} \frac{\partial p(t; x)}{\partial t} = & -\frac{\partial (p(t; x) a_i(t; x))}{\partial x_i} + \frac{1}{2} \frac{\partial^2 (p(t; x) b_{i,k}(t; x) b_{j,k}(t; x))}{\partial x_i \partial x_j} + \\ & + \int [p(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)) \bar{D}(x^{-1}(t; x; \gamma)) - p(t; x)] \Pi(d\gamma). \end{aligned} \quad (33)$$

Equation (33) is *Kolmogorov's equation for density function*.

Now we will define equations for a transition probability density for random processes which are solutions for equations (14).

A density for transition probability for random process is a non-random function  $p(t; x/s; y)$  which provides equality

$$p(t; x) = \int_{\mathbb{R}^n} p(t; x/s; y) p(s; y) d\Gamma(y), \quad t > s. \quad (34)$$

From Eq. (34) we conclude, that the function  $p(t; x/s; y)$  does not depend on a distribution function of  $p(s; y)$  for each  $s \geq 0$ , hence, it is independent with respect to an arbitrary function  $p(0; y) = \rho(y)$ .

After the substitution (34) in Eq. (33) and taking into account an randomness of function  $p(s; y)$  we get that term (34) holds if the next equality is fulfilled

$$\begin{aligned} & \frac{\partial p(t; x/s; y)}{\partial x_i} = \\ & = - \left[ \frac{\partial (p(t; x/s; y) a_i(t; x))}{\partial x_i} - \frac{1}{2} \frac{\partial^2 (p(t; x/s; y) b_{i,k}(t; x) b_{j,k}(t; x))}{\partial x_i \partial x_j} \right] + \\ & + \int \left[ p(t; x - g(t; x^{-1}(t; x; \gamma); \gamma)/s; y) \bar{D}(x^{-1}(t; x; \gamma)) - \right. \\ & \left. - p(t; x/s; y) \right] \Pi(d\gamma). \end{aligned}$$

This is the Kolmogorov forward equation for the transition densities connected with Eq. (14).

Right now let us construct the backward equation for  $p(t; x/s; y)$ .

Since  $p(t; x/s; y)$  is non-random function, we can rewrite Eq. (34) as

$$p(t; x) = \int_{\mathbb{R}^n} \mathbf{M}[p(t; x/s; y) \rho(s; y)] d\Gamma(y).$$

Taking into account this representation and the condition (31) for function  $\rho(s; y)$  we go to the next equality

$$p(t; x) = \int_{\mathbb{R}^n} p(t; x/s; y) \mathbf{M}[\rho(s; y)] d\Gamma(y) = \int_{\mathbb{R}^n} \mathbf{M}[p(t; x/s; y(s; z)) \rho(0; z)] d\Gamma(z),$$

where  $y(s; z)$  is the solution if the Itô equation (14).

Since the left side of this equality does not depend on  $s$  we use the Itô formula with respect to  $s$  and obtain the following

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \mathbf{M}[d_s p(t; x/s; y(s; z)) \rho(0; z)] d\Gamma(z) = \\ &= \int_{\mathbb{R}^n} \mathbf{M} \left\{ \left[ \frac{\partial}{\partial s} p(t; x/s; y(s; z)) + a_j(s; y(s; z)) \frac{\partial}{\partial y_j} p(t; x/s; y(s; z)) + \right. \right. \\ & \quad \left. \left. + \frac{1}{2} b_{j,k}(s; y(s; z)) b_{i,k}(s; y(s; z)) \frac{\partial^2}{\partial y_j \partial y_i} p(t; x/s; y(s; z)) \right] ds + \right. \\ & \quad \left. + b_{j,k}(s; y(s; z)) \frac{\partial}{\partial y_j} p(t; x/s; y(s; z)) dw_k(s) + \right. \\ & \quad \left. + \int [p(t; x/s; y(s; z) + g(t; y(s; z); \gamma)) - p(t; x/s; y(s; z))] \nu(ds; d\gamma) \right\} \rho(0; z) d\Gamma(z). \end{aligned}$$

Taking into account the property (31) again, we get the next expression:

$$\begin{aligned} & \int_{\mathbb{R}^n} \left\{ \left[ \frac{\partial}{\partial s} p(t; x/s; y) + a_j(s; y) \frac{\partial}{\partial y_j} p(t; x/s; y) + \right. \right. \\ & \quad \left. \left. + \frac{1}{2} b_{j,k}(s; y) b_{i,k}(s; y) \frac{\partial^2}{\partial y_j \partial y_i} p(t; x/s; y) \right] + \right. \\ & \quad \left. + \int [p(t; x/s; y + g(s; y; \gamma)) - p(t; x/s; y)] \Pi(d\gamma) \right\} p(s; y) d\Gamma(y) = 0. \end{aligned}$$

The condition that  $p(t; x/s; y)$  does not depend on  $p(s; y)$  is fulfilled if the next result holds

$$\begin{aligned} & \frac{\partial}{\partial s} p(t; x/s; y) + a_j(s; y) \frac{\partial}{\partial y_j} p(t; x/s; y) + \\ & + \frac{1}{2} b_{j,k}(s; y) b_{i,k}(s; y) \frac{\partial^2}{\partial y_j \partial y_i} p(t; x/s; y) + \\ & + \int \left[ p(t; x/s; y + g(s; y; \gamma)) - p(t; x/s; y) \right] \Pi(d\gamma) = 0. \end{aligned}$$

This is *the Kolmogorov backward equation* for the transition densities.

Let us consider Eq. (14). We used the Itô equation with a random (non centered) Poisson measure (14). If we can deal with the Itô equation with a centered Poisson measure, we can use a passage to it Eq. (14).

This approach implies a substitution from coefficients  $a_j(\tau; z)$  to

$$\bar{a}_j(\tau; z) = a_j(\tau; z) - \int g(\tau; z; \gamma) \Pi(d\gamma)$$

and resulting equations are the same as the equations which were obtained by I.I.Gihman and A.V.Skorohod [23].

## 4. Conclusion

The concept of the stochastic kernel function for the stochastic integral invariant allows us to obtain the well-known results such as the Itô – Wentzell formula [5] and the Kolmogorov equations; it demonstrate the correctness of a theory based on this concept. Moreover, the method of the integral invariant gives the chance for developing the theory of the stochastic differential equations and its applications, for example, first integrals and stochastic first integrals, and the program control with probability one (PCP1) [1, 9, 17, 19, 20].

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